The Speed of Light May not be Constant

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In 1911, Einstein demonstrated that a gravitational field could make light slow down by the increase of the index of refraction in vacuum[1]*. On this basis, we formulate the hypothesis that the expansion of the universe could diminish the influence of gravitational fields on light, allowing therefore a slight acceleration of it. According to our estimations, this speed of light (299 792 458 m/s) would increase by 1 m/s every 35.4 years.*

In 1972 and 1973, NASA sent the Pioneer 10/11 probes in space [2] *and noticed a few years later that the probes were slowing down in an unexplained way* [3,4,26,31]*. The Doppler Effect, used to measure the speed of the probes, was taking for granted that the speed of light was constant. This presupposition would have led us to believe, mistakenly according to us, that the probes were slowing down.*

Referring to well-known constants c, G, and H_0, our work proposes 4 equations which will allow us to obtain the ratio β between the speed of expansion of the material universe and the speed of light, the apparent mass of the universe m_u, its radius of curvature r_u, and the asymptotical speed of light k.. These parameters allow the calculation of the acceleration of light $a_L \approx 8.95\times10^{-10}\ m/s^2$ and the one for the Pioneer probes $a_p=-a_L$ (similar to $a_p \approx -8.74\times10^{-10}\ m/s^2$ from Brownstein and Moffat [31] *).*[2, 3]

KEY WORDS: Light, Pioneer, Einstein, relativity, universe, refraction

1. INTRODUCTION

After the discovery of the Pioneer effect and the analysis of different technical possibilities [3,26], NASA concluded that the phenomenon does not fit in with the known laws of physics. Our work led us to believe and formulate the hypothesis that light accelerates over time due to the expansion of the universe.

We will present our model of the universe and its parameters which will enable us to calculate the acceleration of light a_L. To support our hypothesis, we will link it to the Pioneer effect.

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[1] Article of 1911, see [20]. On the basis of the general relativity theory [1], the variation of the index of refraction had to be revised with a factor 2 [21,37]. This theory is confirmed by the discovery of gravitational lens [32,37].

[2] We have adopted the conventional negative sign for an acceleration directed toward the Sun.

[3] Let's mention that recent work that we realized on the universal gravitational constant and on the Hubble constant [33] allowed us to establish that the acceleration of light is more around $9.16903264(1)\times10^{-10}\ m/s^2$.

2. OUR CONCEPTION OF THE UNIVERSE

2.1. Model of the Universe

The material universe can be compared to a balloon in expansion [16,23]. This balloon is in great part filled with “vacuum”. However, “vacuum” is not nothingness since it is full of electromagnetic waves. Even if vacuum seems to have a zero density, we think that a large quantity of vacuum has not necessarily a zero mass. According to NASA, over 95.6 % of the mass of the universe would be in a form that has never been detected in laboratory (dark energy or dark matter) [12].

To receive an electromagnetic wave, we must have an antenna that is about half a wavelength. We believe that most of the photons contained in the vacuum have wavelengths that are not suitable for antennas that can be made on Earth. They seem invisible or non-existent, hence the term "vacuum", suggesting wrongly that there is nothing. That is why we believe the mass and the energy of the vacuum still eludes astronomers. However, their large-scale effects are measurable. The mass and energy of the vacuum have also found their name: dark energy and dark matter.

The expansion of the universe has been confirmed by observing a redshift in the optical spectrum of light coming from distant galaxies [15]. Matter moves away from an enormous empty center [30].The expansion of the immaterial universe (light) is at the speed of light [29]. However, it cannot be the same for the material universe (galaxies, stars, etc.) since it would imply an infinite energy (when $v \rightarrow c$ in the equations of kinetic energy) [25]. Since the universe is still in expansion after many billion years, the speed of expansion must be very high. We will say that the speed of the expansion of the material universe is equal to $\beta \cdot c$. Here, c is the actual speed of light. We will eventually see that this value is not constant as a function of the radius of the universe.

L. Lorenzi thinks that the sphere created by the Big Bang has a center and gives a radius to our actual universe.[4] The observation of a CMB dipole excludes the

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[4] *“[…] center of Big Bang sphere […] of the sphere expanded to R_0 to t_0 […],”* p. 1167 from reference [30].

possibility of a cosmology without center.[5] For an observer positioned at the center of the sphere, the surface of the universe is homogenous [6] and the center of mass is located in the middle of the sphere. All calculations relative to the gravitational potential of the universe may be realized by using the radius r_u and the mass m_u. Many now admit that the universe has a finite mass [13,19] and radius [6], especially since Edwin Hubble has demonstrated that the universe is expanding [22]. Even if there is no consensus on the value that each of these parameters should have, some people present [6,13,19] results similar to ours.

2.2. The Speed of Light and the Speed of Matter as Functions of Time

We will leave it to others to validate the Big Bang theory. We will assume that the light began to accelerate **from the horizon** which is located between the center of mass of the universe and our current position. The universe being in expansion, the index of refraction in the vacuum started to decline. The light has accelerated over time to reach the current speed *c*. This value, which represents the speed of light in vacuum, can be evaluated only in approximation since we cannot stand outside of all gravitational fields. We are always under the influence of some celestial objects.

In 1973, the Evenson team measured the speed of light with the help of a laser and obtained $299\,792\,457.4 \pm 1.2$ m/s [18]. Its value was reevaluated at $299\,792\,458$ m/s in 1983 by the B.I.P.M. [17] and has not been changed since. Based on the eq. **(21)** that we will develop later, the speed of light would have increased by about 1.1 m/s between January 1st of the year 1973 and January 1st of the year 2011. The variation is still within the tolerance of measurement made in 1973 and 1983.

In order to measure a variation 2 times the value of the tolerance on the speed of c, the measuring apparatus must be much more precise than in 1983. If the speed of light measurements obtained in 1973 and 1983 are exact, we can already begin to detect a tendency of the speed to have increased by about 0.3 m/s over this period (see eq. **(21)**). If we add this value to the measurement of *c* in 1973, we understand why, in 1983, the measured value was $299\,792\,458 \pm 1$ m/s.

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[5] *"The observation of the cmb dipole excludes the possibility of a cosmology without center. Thus, there has to be a center for the expansion of the universe, since a CMB dipole has been observed for the solar system,"* p. 3 from reference [16].

[6] *"Unless the observer is positioned exactly at the center of the bubble, the distribution of matter, as seen by the observer, will be anisotropic,"* p. 1 from reference [28].

According to eq. **(21)**, in 2011, the speed of light has increased by about 1 m/s since 1973. This is why we still consider the value measured in 1983 as being correct. Based on the official value measured in 1983, and with the actual resolution of ±1 m/s, we will have to consider a change in the last significant digit of our reference value in about 2054 since light will be 2 m/s faster.

Being measured on Earth, the light velocity is influenced by the gravitational effect of the Sun and the Earth. Thanks to eq. **(9)** that we will see later, it is possible to take into account the gravitational effect of the Sun and the Earth and deduce the real speed of light c in vacuum outside any gravitational field. The constant c is about 6 m/s higher than the speed of light that we measure on Earth.

The light will accelerate over time and we think it will tend toward an asymptotical value that we will name k. One distinguishing item in our work is the introduction of β, which is the ratio between the speed of expansion of the material universe and the speed of light.

3. CALCULATIONS OF THE ACTUAL UNIVERSE PARAMETERS

To obtain the light acceleration a_L, we can represent the universe with the β, k, m_u and r_u parameters which come from the actual values of c and H_0. All these values are "apparent" and probably different from the reality. The universe is huge, massive and very old. Even using the best theories like the relativity of Einstein, there is the risk of obtaining the wrong values which characterize the universe. For example, there is the big temptation of believing that the mass of the universe and the universal gravitational constant G are constant. We wish it, but nothing proves it beyond any doubt. Two options are then offered to us:

1) To search for the true values of the actual parameters of the universe using the relativity of Einstein.
2) To use the "apparent" values of the actual parameters of the universe.

Let's define what we mean by the apparent values. The different parameters of the universe can follow non-linear curves as a function of the radius of the universe. Since the universe is huge and very old, we would be tempted to believe that the different parameters are constant or that they follow a linear progression on a short period of time. These values are what we will call "apparent".

3.1. H_0 : Derivative of the Speed of Matter as a Function of Distance

In 1929, an astronomer, Hubble, noticed that galaxies were moving away from one another with a speed proportional to the distance which separated them [22]. He baptized the proportionality ratio H_0. It represents the variation of the speed of matter per unit of distance. Its value is between 70.4 [12] and 76.9 km/(s·MParsec) [24]. In our document, we will use the value of 70.4 km/(s·MParsec) since it comes from the most recent results of the WMAP project of NASA [12]. [7]

We formulate the hypothesis that the derivative of the expansion speed of the matter with regard to the apparent radius of curvature of the universe, evaluated at r_u, is equal to the actual Hubble constant.

$$\left.\frac{dv_m(r)}{dr}\right|_{r=r_u} = H_0 \tag{1}$$

3.2. Apparent Radius of Curvature of the Universe r_u

The universe is actually expanding at the speed of light c. However, according to the equations of Einstein's relativity, matter (including ourselves) that makes up the universe must travel at a speed v_m which is less than c. Let's suppose that it travels to $v_m=\beta\cdot c$ where $\beta < 1$.

In the derivative described in (1), v_m that is evaluated to r_u is equal to $\beta\cdot c$.

$$\left.\frac{dv_m(r)}{dr}\right|_{r=r_u} = \frac{\beta\cdot c}{r_u} = H_0 \tag{2}$$

So, the apparent radius of curvature of the actual universe is r_u :

$$r_u = \frac{\beta\cdot c}{H_0} \tag{3}$$

3.3. Apparent Mass of the Universe m_u

Let's formulate the hypothesis that the expansion of the universe happened in the

[7] We would like to precise that according to new work that we made, we are now able to calculate precisely the Hubble constant. According to these calculation, its value would rather be around 72.09548580(32) km/(s·MParsec) [33].

same way, that is, in the infinitely small and infinitely large. If our hypothesis is correct, the ratio of the mass and the radius would be the same at the universe level as it is in the smallest possible theoretical particle. To make a proper comparison, it is important to compare the same elements in both cases.

The actual apparent mass of the universe is m_u. The apparent radius of curvature of the universe at our actual position is r_u. However, since light is faster than matter, it went further and the whole universe would rather have an apparent radius of curvature of r_u/β.

The smallest particle would have the theoretical radius of the Planck length L_p. Such a particle would spin on itself at the speed of light c and would have a mass equal to the Plank mass m_p. The mass/radius ratios are:

$$\frac{m_u}{\left(\frac{r_u}{\beta}\right)} = \frac{m_p}{L_p} \quad \textbf{(4)}$$

Knowing that m_p and L_p are defined as being:

$$m_p = \sqrt{\frac{h \cdot c}{2 \cdot \pi \cdot G}} \quad \textbf{(5)}$$

$$L_p = \sqrt{\frac{h \cdot G}{2 \cdot \pi \cdot c^3}} \quad \textbf{(6)}$$

We obtain the following result :

$$\frac{m_u}{r_u} = \frac{m_p}{\beta \cdot L_p} = \frac{c^2}{\beta \cdot G} \quad \textbf{(7)}$$

Using the value of r_u found in (3) in the equation (7), we obtain m_u:

$$m_u = \frac{c^3}{G \cdot H_0} \quad \textbf{(8)}$$

This result is identical to the one obtained by M. Joel C. Carvalho [19]. So, we will use this result as being correct since various other calculation methods yield to the same result.

3.4. The Speed of Light in a Gravitational Field

The Schwarzschild equation, based on the general relativity, enables us to find the speed of light v_c as a function of the gravitational potential ϕ [21,27] :

$$v_c(\phi) = \frac{c}{\sqrt{\dfrac{1-\dfrac{2\cdot\phi}{c^2}}{1+\dfrac{2\cdot\phi}{c^2}}}} \quad \text{where} \quad \phi = -\frac{G\cdot m}{r} \tag{9}$$

v_c = New speed of light as a function of the distance r
r = Distance between the center of mass m and where v_c is evaluated
m = Mass causing the gravitational field
G = $6.673\,84 \times 10^{-11}\ m^3/(kg\cdot s^2)$ = Universal gravitational constant
c = $2\,997\,924\,58\ m/s$ = Speed of light in vacuum

Einstein showed that the speed of light in vacuum was slower in an intense gravitational field. Using equation **(9)**, we see that, on the surface of the Earth, the influence of the Sun on the speed of light is about -6 m/s and the influence of the Earth is only of -0.4 m/s. At the time, the uncertainty on c was greater than the influence of the Earth and Sun and it was justified to make the approximation that the speed of light was c on the surface of the Earth. In 1973, thanks to the laser, the speed of light was measured at ±1 m/s. From equation **(9)** and v_c measured on the Earth, we can calculate the theoretical value of c out of gravitation.

The universe is in expansion [22] and the speed limit, that is, the speed of light, increases as it moves away from the center of mass. In a distant future, the speed of light will have an asymptotical value different from c that we will name k. For the universe, we must replace the constant c by k in the eq. **(9)** to get the speed of light $v_L(r)$:

$$v_L(r) = \frac{k}{n_u(r)} \quad \text{where} \quad n_u(r) = \sqrt{\frac{1+\dfrac{2\cdot G\cdot m}{k^2\cdot r}}{1-\dfrac{2\cdot G\cdot m}{k^2\cdot r}}} \tag{10}$$

For the universe radius $r = r_u$ we know that v_L must be equal to c:

$$v_L(r_u) = \frac{k}{n_u(r_u)} = c \tag{11}$$

3.5. Acceleration of Light and of the Universe

We want to know the acceleration of light $a_L(r_u)$ and of matter $a_m(r_u)$. The immaterial universe is expanding with the speed $v_L(r)$ (see (10)). Matter, which moves more slowly than light, has the speed $v_m(r)=\beta \cdot v_L(r)$. The parameter β represents the ratio of the speed of matter versus the speed of light.

$$v_m(r) = \frac{\beta \cdot k}{n_u(r)} \qquad \textbf{(12)}$$

The derivative of v_m with regard to r, evaluated at r_u, in the eq. **(12)**, gives H_0:

$$H_0 = \left.\frac{dv_m(r)}{dr}\right|_{r=r_u} = \frac{\beta \cdot y \cdot k}{r_u} \cdot \left(\frac{1}{(1+y)\cdot\sqrt{1-y^2}} \right) \quad \text{où} \quad y = \frac{2 \cdot G \cdot m_u}{k^2 \cdot r_u} \qquad \textbf{(13)}$$

3.6. Numerical Evaluation of *k*, *β*, *r_u* and *m_u*

To evaluate the k , β , r_u and m_u values, we need a minimum of 4 equations. Our equation system is based on the eq. **(3)**, **(8)**, **(11)**, and **(13)**. By solving these, we get:

$$\beta = 3 - \sqrt{5} \approx 0.76 \qquad \textbf{(14)}$$

$$k = c \cdot \sqrt{2+\sqrt{5}} \approx 2 \cdot c \approx 6 \times 10^8 \, m/s \qquad \textbf{(15)}$$

$$m_u = \frac{c^3}{G \cdot H_0} \approx 1.8 \times 10^{53} \, kg \qquad \textbf{(16)}$$

$$r_u = \frac{\beta \cdot c}{H_0} \approx 1 \times 10^{26} \, m \qquad \textbf{(17)}$$

We also find that in the eq. **(13)** :

$$y = \frac{\sqrt{5}-1}{2} = \frac{1}{\varphi} \quad \text{where} \quad \varphi = \frac{1+\sqrt{5}}{2} \text{ (golden ratio)} \qquad \textbf{(18)}$$

β is an exact number which depends on no other constant. The value of k depends only on c and is not evolutionary since it represents the asymptotical limit of the speed of light out of gravitation when the radius of the universe will tend toward infinity. The values of m_u and r_u depend unfortunately on H_0, which limits the precision to about 5 %.

Let's compare our results with others. In one of his articles, Mr. Rañada estimates that the radius of the visible universe is 4 200 MParsec (about 1.3×10^{26} m) [6]. Since the ratio of speed β estimated in eq. **(14)** is close to the unity, it follows that the radius of the material universe estimated in eq. **(17)** is substantially the same order of magnitude as the radius of the immaterial universe. The eq. **(16)** which gives the mass of the universe is the same as that of M. Carvalho [19]. As some of our results are similar to those of other authors, our results seem plausible.

3.7. Actual Light Acceleration a_L and Matter Acceleration a_m

Here, we want to calculate a_L et a_m. We anticipate that the amplitude of a_L is equal to the asymptotical value of the Pioneer acceleration. According to eq. **(1)** and from the fact that $v_m(r)=\beta\cdot v_L(r)$, we get :

$$\left.\frac{dv_m(r)}{dr}\right|_{r=r_u}=\left.\frac{d(\beta\cdot v_L(r))}{dr}\right|_{r=r_u}=\left.\frac{\beta\cdot d(v_L(r))}{dr}\right|_{r=r_u}=H_0 \tag{19}$$

$$\left.\frac{d(v_L(r))}{dr}\right|_{r=r_u}=\frac{H_0}{\beta} \tag{20}$$

$$c=\left.\frac{dr}{dt}\right|_{r=r_u} \tag{21}$$

From the equations **(20)** et **(21)**, we obtain the light acceleration $a_L(r_u)$:

$$a_L(r_u)=\left.\frac{dv_L(r)}{dt}\right|_{r=r_u}=\left.\left(\frac{dr}{dt}\cdot\frac{dv_L(r)}{dr}\right)\right|_{r=r_u}=c\cdot\left(\left.\frac{dv_L(r)}{dr}\right|_{r=r_u}\right) \tag{22}$$

$$a_L(r_u)=\frac{c\cdot H_0}{\beta}\approx 8.95\times10^{-10}\,\text{m/s}^2 \tag{23}$$

We see that β is an essential factor to determine the acceleration of light. Now, let's calculate the acceleration of the matter $a_m(r_u)$. The acceleration $a_m(r_u)$ can be calculated from the eq. **(1)** and **(21)** :

$$a_m(r_u)=\left.\frac{dv_m(r)}{dt}\right|_{r=r_u}=\left.\left(\frac{dr}{dt}\cdot\frac{dv_m(r)}{dr}\right)\right|_{r=r_u}=c\cdot H_0\approx 6.84\times10^{-10}\,\text{m/s}^2 \tag{24}$$

As we anticipate that the asymptotical value of the module of the Pioneer acceleration is equal to the light acceleration, the β factor distinguishes our results from those obtained by others [6,7,8]. Taking into account the factor β, our calculations are closer to the value of the Pioneer acceleration measured by NASA (which is -8.74×10^{-10} m/s^2). Let's note that the acceleration of light determined by others is $c \cdot H_0$, which would rather correspond, according to our results, to the acceleration of matter rather than to the acceleration of light.

4. SOLUTION OF THE PIONEER EFFECT

The fact that light accelerates in time gives the feeling that the speed of objects, measured by the Doppler Effect, slows down. This illusion led NASA to believe, wrongly, that there is an acceleration of the Pioneer probes toward the Sun.

To support our hypothesis that light accelerates over time, we will present a complete solution of the Pioneer effect. We will begin by describing this phenomenon. Then we will present possible solutions and assumptions already considered. We will show findings measured according to scale that led us to make a link between the a_L and a_p. We will present the index of refraction of the vacuum. We shall eventually make the calculation of a_p by making the link with the acceleration of light.

4.1. Description of the Pioneer Acceleration a_p

On March 1972 and April 1973, NASA launched the Pioneer 10/11 probes in almost opposite directions [2,26]. In 1980, at 20 au from the Sun, a variation of $5.99 \times 10^{-9}\ Hz/s$ was measured by the Doppler Effect [3] and was interpreted as an acceleration a_p [3,4,5,6,7,26,31] of probes directed toward the Sun [4].

$$a_p \approx -8.74 \pm 1.33 \times 10^{-10}\, m/s^2 \qquad \textbf{(25)}$$

Brownstein and Moffat [31] presented a curve fit illustrating the data collected from the Pioneer 10/11 probes. The value in **(25)** then became the reference in documents dealing with the Pioneer effect [3,4,5,6,7,26,31].

NASA seems to discard technical problems [3,26]. The Galileo and Ulysses probes have also been subjected to an acceleration a_p [5]. It seems possible that the laws of Newton and relativity do not describe the Pioneer effect adequately.

4.2. Avenues of Solutions and Hypotheses

Several have tried to resolve the mystery surrounding the Pioneer effect. Some think that the use of poor references may explain the phenomenon [5]. Others use the MOND theory [9] or involve the dark matter and/or dark energy [10,11].

Some find an explanation in the acceleration of clocks [6]. Many describe the Pioneer effect by $a_p = -c \cdot H_0$ [6,7,8].[8] M. Antonio F. Rañada establishes a direct link between the acceleration of light and the Pioneer acceleration by stating that $a_L = c \cdot H_0 \approx 0.8 \cdot a_p$ [6]. He also predicts an acceleration of clocks.

Until now, no theory explains all the following points in a systematic way:

- $a_p \approx -8.74 \pm 1.33 \times 10^{-10}\ m/s$ after *20 au* directed toward the Sun.
- The acceleration climbs up to *14 au* and later becomes quasi-constant.

We have analyzed the following possibilities without any success:

- The Sun or the Earth could have created a gravitational lens
- A change in the index of refraction in vacuum could affect the Doppler Effect.
- The gain in energy created by the acceleration of light in an expanding universe could have been compensated by a decrease in potential energy.
- The mass-energy of the Sun could increase in time because of the light acceleration in an expanding universe.

Our last attempt, which seems correct, has been to use the hypothesis that light accelerates over time to explain the Pioneer acceleration.

4.3. Observations by Scales of Size

When we multiply H_0 by c, we obtain a_p, with a 28 % error (see eq. **(25)**).

$$a_p \approx -c \cdot H_0 \approx 7.00 \times 10^{-10} m/s^2 \quad \textbf{(26)}$$

This equation lets us believe that a_p is the required light acceleration so that after a time equal to the age of the universe, light in vacuum has the current speed c. The constancy of c, as postulated by Einstein [14], would not be true anymore.

Reconciling eq. **(26)** and eq. **(25)** seems unlikely. A multiplicative factor of about 1,28 would be needed to obtain -8.74×10^{-10} m/s^2. The equation **(26)** comes from comparisons of magnitudes which lead to a close value of the Pioneer acceleration. We have seen at eq. **(23)** that a constant of proportionality is missing in eq. **(26)** and that there is a direct link between a_p and a_L :

$$a_p = -a_L = -\frac{c \cdot H_0}{\beta} \approx -8.95 \times 10^{-10} m/s^2 \quad \textbf{(27)}$$

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[8] *We used a negative sign to show that the acceleration is directed toward the Sun.*

a_L does not implicitly explain that Pioneer 10/11 decelerate. a_L and a_p have the same amplitude, but not the same sign. Furthermore, if the data coming from Pioneer 10/11 are exact, a_p depends on the distance Sun-probe.

4.4. Refraction Index in Vacuum

In **(9)**, if $2{\cdot}G{\cdot}m / (c^2 \cdot r) << 1$ (true for the solar system):

$$n(r) \approx 1 + \frac{2 \cdot G \cdot m}{c^2 \cdot r} \tag{28}$$

When a given area is under the influence of many gravitational fields, it is as if we had many masses in superposition:

$$n_{total} \approx 1 + \frac{2 \cdot G \cdot (m_1 + m_2 + ... + m_i)}{c^2 \cdot r} = 1 + \frac{2 \cdot G \cdot m_1}{c^2 \cdot r} + \frac{2 \cdot G \cdot m_2}{c^2 \cdot r} + ... + \frac{2 \cdot G \cdot m_i}{c^2 \cdot r} \tag{29}$$

It is equivalent to making the summation of the variations of index of refraction:

$$n_{total} \approx 1 + (n_1 - 1) + (n_2 - 1) + ... + (n_i - 1) = 1 + \Delta n_1 + \Delta n_2 + ... + \Delta n_i \tag{30}$$

When the index of refraction variations are small compared with unity, the following approximation is valid:

$$n_{total} \approx n_1 \cdot n_2 \cdot n_3 \cdot n_4 \cdot ... \cdot n_i \tag{31}$$

Let's make the demonstration.

$$n_1 \cdot n_2 \cdot n_3 \cdot n_4 \cdot ... \cdot n_i = (1 + \Delta n_1) \cdot (1 + \Delta n_2) \cdot (1 + \Delta n_3) \cdot (1 + \Delta n_4) \cdot ... \cdot (1 + \Delta n_i) \tag{32}$$

If we take the first bracket and multiply it by the second one, we get:

$$n_{total} \approx (1 + \Delta n_1 + \Delta n_2 + \underbrace{\Delta n_1 \cdot \Delta n_2}_{negligible}) \cdot (1 + \Delta n_3) \cdot (1 + \Delta n_4) \cdot ... \cdot (1 + \Delta n_i) \tag{33}$$

When the index of refraction variations are very small compared with unity, the product of the index of refraction variations is negligible:

$$n_{total} \approx (1 + \Delta n_1 + \Delta n_2) \cdot (1 + \Delta n_3) \cdot (1 + \Delta n_4) \cdot ... \cdot (1 + \Delta n_i) \tag{34}$$

If we take the first bracket in the eq. (34) and multiply it by the second one, still neglecting the products of the index of refraction variations, we get:

$$n_{total} \approx (1 + \Delta n_1 + \Delta n_2 + \Delta n_3) \cdot (1 + \Delta n_4) \cdot ... \cdot (1 + \Delta n_i) \tag{35}$$

Using the same process until $(1+\Delta n_i)$, we conclude that:

$$n_{total} \approx 1 + \Delta n_1 + \Delta n_2 + \Delta n_3 + ... + \Delta n_i \approx n_1 \cdot n_2 \cdot n_3 \cdot n_4 \cdot ... \cdot n_i \tag{36}$$

For simplicity, when calculations involving different relatively weak gravitational fields are performed, it is preferable to use the approximation shown in **(31)**.

4.5. Calculation of the Pioneer Effect

The normal gravitational effect of the Sun slows down the probes which move away from the Sun, but we will not take this into account to show the Pioneer acceleration. Now, let's suppose a probe that leaves the Earth with a constant speed v_p. NASA measures its speed with the Doppler Effect [14]. The probe always emits the frequency $f_0=c/\lambda_0$ (where λ_0 is the wavelength):

$$f = f_0 \cdot \sqrt{\frac{1-\frac{v_p}{c}}{1+\frac{v_p}{c}}} = \frac{c}{\lambda_0}\sqrt{\frac{1-\frac{v_p}{c}}{1+\frac{v_p}{c}}} \qquad \textbf{(37)}$$

For an observer at a standstill with regard to the object, the perceived frequency will be f rather than f_0. Knowing f_0, the speed v_p can be calculated by measuring the frequency f.

Unlike an experiment made on Earth, the Pioneer probes move away from the Earth and Sun. They travel through a considerable distance and for a long period of time. We must then consider that the light accelerates over time.

From a mathematical point of view, there are 2 different but equivalent ways of interpreting the variation of f over the time, measured by NASA:

1. We continue to think that c is constant and this leads us to conclude that there is a Pioneer acceleration.
2. There is no Pioneer effect, but an acceleration of light.

Let's analyze the first option. Let's suppose that the probe makes a journey lasting Δt, away from the Sun. Δt can be calculated as a function of r relatively to the Sun, knowing that the distance Sun-Earth is r_{st} (1 au):

$$\Delta t = \frac{r - r_{ST}}{v_p} \quad \text{for } r \geq r_{ST} \qquad \textbf{(38)}$$

The acceleration of the light gives the illusion that the probes have an acceleration $a_p(r)$ and a speed reduction of $a_p(r)\cdot\Delta t$. Putting the variation of speed positive in the eq. **(39)** will put the sign of the Pioneer acceleration in evidence. A negative sign will mean that it is directed toward the Sun since the biggest variation of the refraction index comes from it. From **(37)** and **(38)**:

$$f(r)=\frac{c}{\lambda_0}\cdot\sqrt{\frac{1-\frac{\left(v_p+a_p\cdot\frac{(r-r_{ST})}{v_p}\right)}{c}}{1+\frac{\left(v_p+a_p\cdot\frac{(r-r_{ST})}{v_p}\right)}{c}}} \tag{39}$$

Now, let's analyze the second option: the expansion of the universe forces the light to accelerate over the time. The constant c in the eq. **(37)** must then be replaced by the function $v_L(\Delta t)$ which will take a_L into account (see eq. **(23)**):

$$v_L(\Delta t)=c+a_L\cdot\Delta t=c\cdot\left(1+\frac{H_0\cdot\Delta t}{\beta}\right)\approx c\cdot\left(1+\frac{H_0}{\beta}\cdot\left(\frac{r-r_{ST}}{v_p}\right)\right) \tag{40}$$

This equation takes into account only the light acceleration due to the expansion of the universe during a journey of duration Δt.

By moving away from the Sun, the refraction index in vacuum decreases and causes the speed of light to increase. Since the observers on Earth think that the speed of light is constant, they are not aware that the reference frequency of their measuring instruments changes over time. If, on Earth, we do not see any variation over time when we compare two clocks having the same frequency, we will think that they keep the same frequency. However, in reality, all clocks will have accelerated with the same value over the time.

We must apply a multiplicative factor to the second part of the first parenthesis of **(40)**. This factor must be null on Earth and must be equal to unity when the implied distances are infinite. Between the two, the factor must change as the refraction index of the vacuum changes since it is this change that causes the Pioneer effect. On the surface of the Earth, the two predominant influences are the Sun and the Earth. So the total refraction index on the Earth surface is $n_S(r_{ST})\cdot n_T(r_T)$. Here, r_{ST} is the distance Sun-Earth and r_T is the Earth radius. The difference between the total refraction index at infinity and the refraction index on the Earth's surface is given by Δn_{Total}:

$$\Delta n_{Total}=n_S(\infty)\cdot n_T(\infty)-n_S(r_{ST})\cdot n_T(r_T) \tag{41}$$

When $r\to\infty$, there is no more gravitational influence and the index of refraction tends toward unity. So the equation (49) can be rewritten like this:

$$\Delta n_{Total}=1-n_S(r_{ST})\cdot n_T(r_T) \tag{42}$$

Now, let's determine the variation of refraction index that an object can have by moving away from the Earth. Let's name this variation $\Delta n(r)$; it is a function of the distance r with respect to the Sun and is given by:

$$\Delta n(r) = n_S(r) \cdot n_T(r - r_{ST} + r_T) - n_S(r_{ST}) \cdot n_T(r_T) \qquad \textbf{(43)}$$

Let's note that right from the beginning, the refraction index variation *Δn(r)* is null on the Earth's surface. It becomes equal to Δn_{Total} when *r* tends toward infinity. The ratio $\Delta n(r) / \Delta n_{Total}$ enables us to know in what proportion the total refraction index caused by the Sun and the Earth may be felt when the probes move away from the Earth and the Sun. The same ratio will multiply the second term of **(40)** to give this:

$$v_L(r) = c \cdot \left(1 + \frac{H_0}{\beta} \cdot \left(\frac{r - r_{ST}}{v_P}\right) \cdot \left(\frac{\Delta n(r)}{\Delta n_{Total}}\right)\right) \qquad \textbf{(44)}$$

We can rewrite **(37)** as follows by replacing *c* by v_L:

$$f = \frac{v_L}{\lambda_0} \cdot \sqrt{\frac{1 - \frac{v_p}{v_L}}{1 + \frac{v_p}{v_L}}} \qquad \textbf{(45)}$$

The two ways of observing the situation give the same result. Let's make the eq. **(39)** and **(45)** equal. By solving the equation to find $a_p(r)$, we obtain:

$$a_p(r) = -v_p \cdot \frac{\left(c \cdot v_L^3 - v_p \cdot c \cdot v_L^2 - c^3 \cdot v_L - c^3 \cdot v_p + v_p \cdot v_L^3 - v_p^2 \cdot v_L^2 + v_p \cdot c^2 \cdot v_L + c^2 \cdot v_p^2\right)}{\left(v_L^3 - v_p \cdot v_L^2 + c^2 \cdot v_L + c^2 \cdot v_p\right) \cdot (r - r_{ST})} \qquad \textbf{(46)}$$

Since $v_L \approx c$ and $v_p << c$, we obtain the following approximation by neglecting the terms containing v_p in the brackets of **(46)**.

$$a_p(r) \approx \frac{-v_p \cdot c \cdot \left(v_L^2 - c^2\right)}{(r - r_{ST}) \cdot \left(v_L^2 + c^2\right)} \qquad \textbf{(47)}$$

By replacing v_L by eq. **(44)**, we obtain:

$$a_p(r) \approx \frac{-v_p \cdot c}{(r - r_{ST})} \cdot \left[\frac{\left[1 + \frac{H_0}{\beta} \cdot \left(\frac{r - r_{ST}}{v_p}\right) \cdot \left(\frac{\Delta n(r)}{\Delta n_{Total}}\right)\right]^2 - 1}{\left[1 + \frac{H_0}{\beta} \cdot \left(\frac{r - r_{ST}}{v_p}\right) \cdot \left(\frac{\Delta n(r)}{\Delta n_{Total}}\right)\right]^2 + 1}\right] \qquad \textbf{(48)}$$

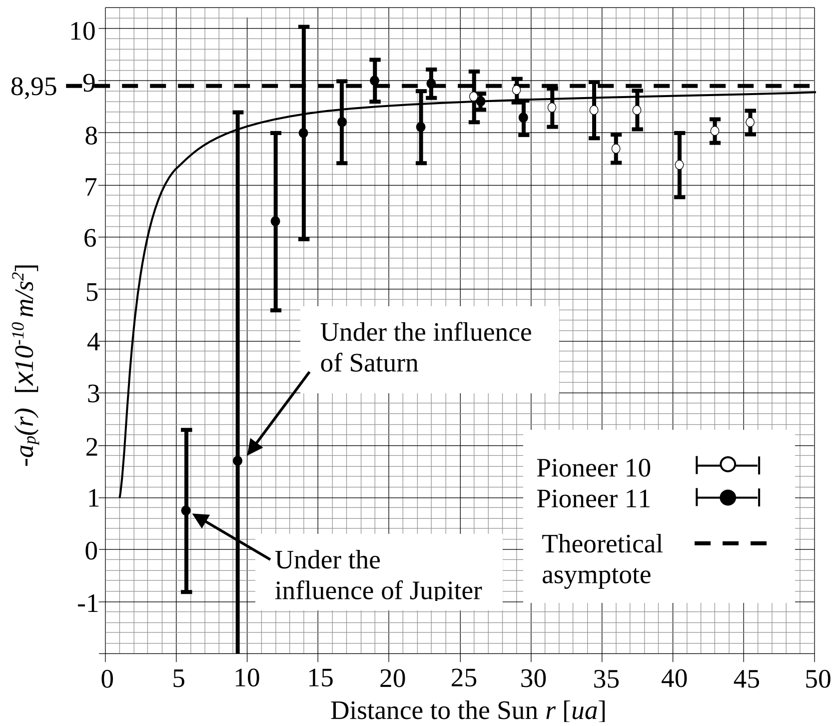

Figure 1 The curve of $-a_p(r)$ comes from eq. **(49)**. Here, we did not take into account the influence of Jupiter and Saturn on the theoretical curve of the Pioneer effect. The acceleration shown for Pioneer 10/11 comes from Brownstein and Moffat [31].

By doing a few other approximations, we obtain:

$$a_p(r) \approx -\frac{c \cdot H_0}{\beta} \cdot \left(\frac{\Delta n(r)}{\Delta n_{Total}} \right) \tag{49}$$

In (49), if $r = r_{ST}$ (on Earth), $a_p = 0$ (see (42) and (43)). But, if $r = \infty$, a_p is :

$$a_p \approx -\frac{c \cdot H_0}{\beta} \approx -8.95 \times 10^{-10}\, m/s^2 \tag{50}$$

In eq. **(49)** and **(50)**, a_p is independent of the speed of the probes. Contrary to a_L, a_p takes a negative sign. The negative sign means that the acceleration is pointed toward the Sun and opposes its action against the movement of the probe which has a speed v_p as it moves away from the Sun. The module of the asymptotical value of a_p is exactly equal to a_L .

In Figure 1, the experimental data shown between *1 au* and *10 au* were taken close to Jupiter and Saturn. They strongly tended to decline from the increase of the refraction index of the vacuum close to these planets. The eq. **(49)** does not take into account the fact that the Pioneer probes are passing close to planets. However, by modifying **(42)** and **(43)** with the help of eq. **(31)**, it would be relatively easy to take the influence of the other planets into account. We would obtain a doubly saw-toothed curve which would fit with the data collected by NASA around Jupiter and Saturn. We do not show this curve since the exact distances of approach with respect to the planets are not known to us.

The fact that we consider the speed of light to be constant gives the impression that the probes slow down. But, in reality, this is not what is happening! The probes do not slow down. It is the light that accelerates and creates the Pioneer effect.

5. A FEW POSSIBLE IMPACTS OF THE WORK

5.1. Acceleration of Clocks

One of the consequences of not taking the acceleration of light over time into account is to "create" an acceleration of time. In one his articles, Mr. Rañada shows that clocks (of all types) accelerate over time [6].

The « second », as defined in 1968 in the metric system, is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between two superfine levels of the fundamental state of the cesium atom 133 at 0 °K [32].

Let's suppose, as a first step, that the speed of light is constant and equal to *c*. Light will travel a distance *d* during a length of time Δt_1.

$$\Delta t_1 = \frac{d}{c} \tag{51}$$

If light has the acceleration a_L predicted in **(23)**, the distance *d* will be traveled in a length of time Δt_2 (inferior to Δt_1).

$$\Delta t_2 = \frac{d}{c \cdot \left(1 + \frac{H_0 \cdot \Delta t_2}{\beta}\right)} \tag{52}$$

Let's define the time acceleration a_T as being the variation of time per second. The a_T units are seconds/second. Thus, we have:

$$a_T = \frac{\Delta t_2 - \Delta t_1}{\Delta t_1} \tag{53}$$

For small Δt_2 values, the following approximation is acceptable:

$$a_T \approx -\frac{H_0 \cdot \Delta t_2}{\beta} \tag{54}$$

For $\Delta t_2 \approx \Delta t_1$ = 1 second, we obtain the following time acceleration:

$$a_T \approx -\frac{H_0}{\beta} \approx -2.99 \times 10^{-18}\, s/s \tag{55}$$

So, if the light acceleration is $a_L = c \cdot H_0 / \beta \approx 8.95\times10^{-10}$ m/s^2 (see **(23)**), we lose about $H_0 / \beta \approx 2.99\times10^{-18}$ seconds in each second.

The Anderson team measured a constant acceleration of Pioneer 10/11 [3] (see eq. **(25)**). Rañada interprets this value as being an acceleration of clocks:

$$a_T = \frac{a_p \cdot \Delta t}{c} \approx -2.9 \pm 0.3 \times 10^{-18}\, s/s \quad \text{si } \Delta t = 1 \text{ second (from Anderson)} \tag{56}$$

Our theoretical result from **(55)** is 3% lower than the one from **(56)**.

5.2. True Value of H_0

Many equations in astrophysics depend on the value of H_0. From the eq. **(14)**, **(23)**, and **(50)** we can calculate its value from known constants:

$$H_0 = \frac{a_L \cdot \beta}{c} = \frac{-a_p \cdot \beta}{c} \tag{57}$$

Using the value of $a_p \approx -8.74 \pm 1.33 \times 10^{-10}$ m/s^2 from Anderson [3]:

$$H_0 = 68.7 \pm 10.5 \frac{km}{MP \sec} \tag{58}$$

The error scale is still very important, but it corresponds quite well to what is now known about H_0 which is somewhere between *70.4 km/(s·MParsec)* [12] and 76.9 *km/(s·MParsec)* [24]. By using new technologies and probes to measure a_p, we could improve the measurement of H_0.

6. CONCLUSION

6.1. Reaching the Initial Goal

Modern physics teaches us that the speed of light in vacuum is constant and all the instruments of measurement are designed on the basis of this postulate.

After the presentation of our model of the universe, we have seen that the expansion of the universe leads to a decrease of the refraction index in vacuum. This enables light to accelerate over time. According to eq. **(23)**, light, in 2054, will be 2 m/s faster than in 1983.

In a second step, to support the hypothesis of the acceleration of light over time, we made the link with the Pioneer effect. Our theoretical calculations correspond with what has been measured by NASA. So, our hypothesis seems plausible. To make an experience of long duration precise (space travels), we must consider a_L over time.

6.2. Limitations of the Work

We limited ourselves to demonstrate that the speed of light was not constant over time and that this explained the Pioneer effect. For calculation purposes, it was necessary to evaluate certain parameters of our universe (m_u, r_u, k and β) by limiting their use only within the framework of our project.

6.3. Questions Raised for the Future

Our work raises questions which will need studies at greater depths:

- The analysis, using the Doppler Effect, of the velocity of movement of stars and galaxies may be biased by the fact that we thought that the speed of light in vacuum was constant and unchanging throughout the universe.
- The acceleration of light has probably many implications at the atomic level. It would be interesting to discover them.
- It would be interesting to know the implications of our work on the other constants of physics since it may be possible that some of them are being affected by a_L.

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8. APPENDIX A: RESOLUTION OF THE EQUATION SYSTEM Claude Mercier eng. (Added on October 20th, 2018)

In this appendix, we will focus on explaining in detail the calculations that made it possible to solve the system of four equations with four unknowns.

It is generally recognized by astrophysicists [34,35,36] that the apparent radius of curvature of the luminous universe is R_u. This radius is the distance that the light would have traveled during a time $T = 1/H_0$ at a constant speed c.

$$R_u = \frac{c}{H_0} \tag{59}$$

It is true that according to our hypotheses, the speed of light is not constant over time and that it accelerates. This is why we say that it is the "apparent" radius of curvature of the luminous universe.

Currently, without knowing if the universe is expanding or not, without knowing if the speed of light is constant or not, the universe seems to have a radius R_u when we make the following assumptions:

1) That the speed of light in a vacuum is constant and equal to c.
2) That the apparent age of the universe is $T = 1/H_0$.

Assuming we move in the universe at a slower speed of light in vacuum (Einstein showed that it was impossible for a mass to travel at a speed equal to or greater than that of light in vacuum), we are now positioned at a distance r_u which is a fraction of R_u , that to say the arbitrary value $\beta\cdot R_u$. It remains to determine the value of β.

The distance between the center of mass of the universe and where we are now has the value of r_u.

$$r_u = \frac{\beta \cdot c}{H_0} = \beta \cdot R_u \tag{60}$$

In this document, we take for granted that the apparent mass of the universe given by the Carvalho equation [19] is correct:

$$m_u = \frac{c^3}{G \cdot H_0} \tag{61}$$

Using general relativity, Schwarzschild [21,27]showed that the velocity of light v_L in vacuum could be influenced by a gravitational field caused by a large mass when evaluated at a distance r.

$$v_L(r) = \frac{c}{n(r)} \quad \text{where} \quad n(r) = \sqrt{\frac{1 + \frac{2 \cdot G \cdot m}{c^2 \cdot r}}{1 - \frac{2 \cdot G \cdot m}{c^2 \cdot r}}} \qquad \textbf{(62)}$$

Of course, in this equation, when we stretch r to infinity, that is, for a distance that is out of the gravitational field, the speed of light v_L becomes equal to c.

Note that it is with equation **(62)** that it is possible to calculate the radius of the horizon of a black hole. Indeed, on the horizon of a black hole, that is to say at the position r_{TN}, the refractive index $n(r) \to \infty$ and the speed of light is zero. For this precise condition, the denominator of $n(r)$ is equal to 0. We then obtain the equation **(63)** which determines the radius r_{TN} of the horizon of a black hole as a function of its mass m, of the velocity light in the vacuum c (out of local gravitation) and the universal gravitational constant G:

$$r_{TN} = \frac{2 \cdot G \cdot m}{c^2} \qquad \textbf{(63)}$$

To return to equation **(62)**, it is only valid for a part of the universe small enough that the velocity of light v_L out of gravitation is constant and equal to c. As described in our article, the universe is immense and the speed of light is influenced by the gravitational field caused by the mass of the universe m_u. Of course, the asymptotic speed of light off gravitation is no longer c, but something else that we will arbitrarily call k. Even if the final speed of light out of gravitation is no longer the same, the shape of the equation remains the same as in equation **(62)**:

$$v_L(r) = \frac{k}{n_u(r)} \quad \text{where} \quad n_u(r) = \sqrt{\frac{1 + \frac{2 \cdot G \cdot m_u}{k^2 \cdot r}}{1 - \frac{2 \cdot G \cdot m_u}{k^2 \cdot r}}} \qquad \textbf{(64)}$$

For the actual radius of curvature of the universe $r = r_u$, we know that the speed of light v_L must be equal to c since this is what we measure here, in our location in the universe.

So, we have:

$$v_L(r_u) = \frac{k}{\sqrt{\dfrac{1+\dfrac{2\cdot G\cdot m_u}{k^2\cdot r_u}}{1-\dfrac{2\cdot G\cdot m_u}{k^2\cdot r_u}}}} = \frac{k}{\sqrt{\dfrac{1+y}{1-y}}} = c \quad \text{where} \quad y = \frac{2\cdot G\cdot m_u}{k^2\cdot r_u} \tag{65}$$

Equation **(64)** allows finding the speed of light for a distance r with respect with the center of mass of the universe. We cannot see the evolution of the luminous universe. However, when we look at sky, we see stars, and these move at a lower speed than light. In average, the material universe is expanding at a speed $v_m(r)$.

$$v_m(r) = \beta\cdot v_L(r) = \frac{\beta\cdot k}{\sqrt{\dfrac{1+\dfrac{2\cdot G\cdot m_u}{k^2\cdot r}}{1-\dfrac{2\cdot G\cdot m_u}{k^2\cdot r}}}} \tag{66}$$

If we derivate the speed v_m with respect to distance r, we obtain:

$$\frac{dv_m(r)}{dr} = \frac{\beta\cdot y\cdot k}{r}\cdot\left(\frac{1}{(1+y)\cdot\sqrt{1-y^2}}\right) \quad \text{where} \quad y = \frac{2\cdot G\cdot m_u}{k^2\cdot r} \tag{67}$$

For a distance $r = r_u$, this derivate is equal to the Hubble constant H_0 since we obtain the apparent age of the universe evaluated here, at our location.

$$H_0 = \left.\frac{dv_m(r)}{dr}\right|_{r=r_u} = \frac{\beta\cdot y\cdot k}{r_u}\cdot\left(\frac{1}{(1+y)\cdot\sqrt{1-y^2}}\right) \quad \text{where } y = \frac{2\cdot G\cdot m_u}{k^2\cdot r_u} \tag{68}$$

So let's remember the following four equations: **(60)**, **(61)**, **(65)** and **(68)**. In these equations, the unknown values are: r_u, m_u, β and k. It is a system made of 4 equations and 4 unknowns which can theoretically be solved mathematically.

Let's begin the resolution of the equation system.

In equation **(65)**, we isolate the *k* value to obtain:

$$k = c \cdot \sqrt{\frac{1+y}{1-y}} \tag{69}$$

In equation **(68)**, we also isolate the *k* value to obtain:

$$k = \frac{H_0 \cdot r_u}{y \cdot \beta} \cdot (1+y)\sqrt{1-y^2} \tag{70}$$

We make the **(69)** and **(70)** equations being equal to obtain:

$$c \cdot \sqrt{\frac{1+y}{1-y}} = \frac{H_0 \cdot r_u}{y \cdot \beta} \cdot (1+y)\sqrt{1-y^2} \tag{71}$$

Let's square each side of the equation:

$$c^2 \cdot \frac{(1+y)}{(1-y)} = \frac{H_0^2 \cdot r_u^2}{y^2 \cdot \beta^2} \cdot (1+y)^2 \cdot (1-y^2) \tag{72}$$

Let's isolate the square of the speed of light c^2:

$$c^2 = \frac{H_0^2 \cdot r_u^2}{y^2 \cdot \beta^2} \cdot \left(1-y^2\right)^2 \tag{73}$$

From equation **(60)**, we obtain:

$$\frac{r_u^2}{\beta^2} = \frac{c^2}{H_0^2} \tag{74}$$

Let's make the replacement in equation **(73)**:

$$c^2 = \frac{H_0^2 \cdot c^2}{y^2 \cdot H_0^2} \cdot \left(1-y^2\right)^2 \tag{75}$$

Let's make a few simplifications:

$$y^2 = \left(1-y^2\right)^2 \tag{76}$$

Let's make the square root each side of the equation and let's move everything on the left side of the equation:

$$y^2 + y - 1 = 0 \tag{77}$$

Let's isolate y to obtain:

$$y = \frac{-1 \pm \sqrt{5}}{2} \tag{78}$$

Because of equation **(68)**, we also know that:

$$y = \frac{2G \cdot m_u}{k^2 \cdot r_u} \tag{79}$$

All parameters of this equation are strictly positive. Consequently, the y value is necessarily positive:

$$y = \frac{\sqrt{5} - 1}{2} \tag{80}$$

Putting the result of **(80)** in equation **(65)**, we obtain:

$$\frac{k}{\sqrt{\dfrac{1 + \dfrac{\sqrt{5}-1}{2}}{1 - \dfrac{\sqrt{5}-1}{2}}}} = c \tag{81}$$

After simplifications, we obtain:

$$k = c \cdot \sqrt{2 + \sqrt{5}} \tag{82}$$

In equation **(68)**, let's substitute the y value in front the bracket by its algebraic value**:**

$$H_0 = \frac{2G \cdot m_u \cdot \beta}{k \cdot r_u^2} \cdot \left(\frac{1}{(1+y) \cdot \sqrt{1-y^2}} \right) \tag{83}$$

In equation **(83)**, let's replace the m_u value by equation **(61)**:

$$H_0 = \frac{2c^3 \cdot \beta}{k \cdot r_u^2 \cdot H_0} \cdot \left(\frac{1}{(1+y) \cdot \sqrt{1-y^2}} \right) \tag{84}$$

In equation **(84)**, let's replace the r_u value by equation **(60)**:

$$H_0 = \frac{2c^3 \cdot \beta \cdot H_0^2}{k \cdot \beta^2 \cdot c^2 \cdot H_0} \cdot \left(\frac{1}{(1+y) \cdot \sqrt{1-y^2}} \right) \tag{85}$$

Let's simplify and let's isolate β:

$$\beta = \frac{2c}{k} \cdot \left(\frac{1}{(1+y) \cdot \sqrt{1-y^2}} \right) \tag{86}$$

In equation **(86)**, let's replace the k value by equation **(82)** and let's simplify:

$$\beta = \frac{2}{\sqrt{2+\sqrt{5}}} \cdot \left(\frac{1}{(1+y) \cdot \sqrt{1-y^2}} \right) \tag{87}$$

In equation **(87)**, let's replace y by equation **(80)**:

$$\beta = \frac{2}{\left(1 + \frac{\sqrt{5}-1}{2}\right) \cdot \sqrt{2+\sqrt{5}} \cdot \sqrt{1 - \left(\frac{\sqrt{5}-1}{2}\right)^2}} \tag{88}$$

$$\beta = \frac{2}{\left(\frac{2+\sqrt{5}-1}{2}\right) \cdot \sqrt{2+\sqrt{5}} \cdot \frac{\sqrt{4-\left(\sqrt{5}-1\right)^2}}{2}} \tag{89}$$

$$\beta = \frac{8}{(1+\sqrt{5}) \cdot \sqrt{2+\sqrt{5}} \cdot \sqrt{4-(6-2\sqrt{5})}} \tag{90}$$

$$\beta = \frac{8}{(1+\sqrt{5}) \cdot \sqrt{2+\sqrt{5}} \cdot \sqrt{2(\sqrt{5}-1)}} \tag{91}$$

Let's square each side of the equation:

$$\beta^2 = \frac{64}{(1+2\sqrt{5}+5) \cdot (2+\sqrt{5}) \cdot 2(\sqrt{5}-1)} \tag{92}$$

$$\beta^2 = \frac{16}{(3+\sqrt{5}) \cdot (3+\sqrt{5})} \tag{93}$$

$$\beta^2 = \frac{4^2}{(3+\sqrt{5})^2} \tag{94}$$

Let's make the square root each side and let's keep only the positive root since we defined β as being a positive ratio:

$$\beta = \frac{4}{3+\sqrt{5}} \tag{95}$$

Let's multiply the numerator and the denominator by the conjugate of the denominator and let's simplify:

$$\beta = \frac{4}{3+\sqrt{5}} \cdot \frac{3-\sqrt{5}}{3-\sqrt{5}} = 3-\sqrt{5} \tag{96}$$

The β value is unitless.

In summary, we obtain:

$$\beta = 3-\sqrt{5} \approx 0.76 \tag{97}$$

$$k = c \cdot \sqrt{2+\sqrt{5}} \approx 2 \cdot c \approx 6 \times 10^8 \, m/s \tag{98}$$

$$m_u = \frac{c^3}{G \cdot H_0} \approx 1.8 \times 10^{53} \, kg \tag{99}$$

$$r_u = \frac{\beta \cdot c}{H_0} \approx 1 \times 10^{26} \, m \tag{100}$$

We note that it is possible to solve the system of 4 equations and 4 unknowns that we had at the beginning.

Knowing the r_u and β values, it is possible to evaluate the value of the apparent radius of curvature of the universe R_u [34,35,36]:

$$R_u = \frac{c}{H_0} \approx 1.3 \times 10^{26} \, m \tag{101}$$

It should be noted that the reader will be able to note, by reading other more recent documents of the author, that the β value is useful for calculating a multitude of fundamental physics constants.

So, it seems that there is a kind of geometry in matter that connects the infinitely small with the infinitely large.